

## Finite $N$ effects on the collapse of fuzzy spheres

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**ABSTRACT:** Finite  $N$  effects on the time evolution of fuzzy 2-spheres moving in flat space-time are studied using the non-Abelian DBI action for  $N$   $D0$ -branes. Constancy of the speed of light leads to a definition of the physical radius in terms of symmetrised traces of large powers of Lie algebra generators. These traces, which determine the dynamics at finite  $N$ , have a surprisingly simple form. The energy function is given by a quotient of a free multi-particle system, where the dynamics of the individual particles are related by a simple scaling of space and time. We show that exotic bounces of the kind seen in the  $1/N$  expansion do not exist at finite  $N$ . The dependence of the time of collapse on  $N$  is not monotonic. The time-dependent brane acts as a source for gravity which, in a region of parameter space, violates the dominant energy condition. We find regimes, involving both slowly collapsing and rapidly collapsing branes, where higher derivative corrections to the DBI action can be neglected. We propose some generalised symmetrised trace formulae for higher dimensional fuzzy spheres and observe an application to  $D$ -brane charge calculations.

**KEYWORDS:** D-branes, M(atric) Theories, Non-Commutative Geometry.

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**1. Introduction**

The symmetrised trace prescription for the non-Abelian action of multiple  $D0$ -branes was proposed in [1] and extended to include background RR fluxes in [2]. An interesting time dependent system, in which the need for an exact prescription arises, is a spherical bound state of  $N$   $D0$ -branes with a spherical  $D2$ -brane, for finite values of  $N$ . This can be studied both from the point of view of the Abelian  $D2$  DBI action and the non-Abelian  $D0$ -DBI action. The latter configuration also has an M-theory analogue, that of a time dependent spherical  $M2$ -brane, which has been studied in the context of matrix theory [3, 4]. In [5] it was shown that the  $D0$ -brane construction, based on the fuzzy 2-sphere, agrees with the Abelian  $D2$ -construction at large  $N$ .  $1/N$  corrections coming from the symmetrised trace and a finite  $N$  example were also studied. Here we develop further the study of finite  $N$ .

The need for the non-linear DBI action, as opposed to the Yang-Mills limit of the lower dimensional brane, was recognised in a spatial  $D1 \perp D3$  analogue of the  $D0 - D2$  system. The static BIon  $D1 \perp D3$  intersection, in which the  $D1$ 's blow up into a funnel of fuzzy- $S^2$  cross-section, and its higher dimensional generalisations have been studied extensively in the literature [6–10].

In this paper we extend the calculation of symmetrised traces from the spin half example of [5] to general representations of  $SO(3)$ . These results allow us to study in detail the finite  $N$  physics of the time-dependent fuzzy two-sphere. We begin our finite  $N$  analysis with a careful discussion on how to extract the physical radius from the matrices of the non-Abelian ansatz. The standard formula used in the Myers effect is  $R^2 = \lambda^2 \text{Tr}(\Phi_i \Phi_i)/N$ . Requiring consistency with a constant speed of light, independent of  $N$ , leads us to propose an equation in section 2, which agrees with the standard formula in large  $N$  commutative limits, but disagrees in general. Section 3 gives finite  $N$  formulae for the energy and Lagrangian of the time-dependent fuzzy 2-sphere. We also give the conserved pressure which is relevant for the  $D1 \perp D3$  system. In section 4, we study the time of collapse as a function of  $N$ . In the region of large  $N$ , for fixed initial radius  $R_0$ , the time decreases as  $N$  decreases. However, at some point there is a turn-around in this trend and the time of collapse for spin half is actually larger than at large  $N$ . We also investigate the quantity  $E^2 - p^2$ , where  $E$  is the energy and  $p$  the momentum. This quantity is of interest when we view the time-dependent  $D$ -brane as a source for spacetime fields.  $E$  is the  $T^{00}$  component of the stress tensor, and  $p$  is the  $T^{0r}$  component as we show by a generalisation of arguments previously used in the context of BFSS matrix theory. For the large  $N$  formulae,  $E^2 - p^2$  is always positive. At finite  $N$ , this can be negative, although the speed of radial motion is less than the speed of light. Given the relation to the stress tensor, we can interpret this as a violation of the dominant energy condition. The other object of interest is the proper acceleration along the trajectory of a collapsing  $D2$ -brane. We find analytic and numerical evidence that there are regions of both large  $R$  and small  $R$ , with small and relativistic velocities respectively, where the proper accelerations can be small. This is intriguing since the introduction of stringy and higher derivative effects in the small velocity region can be done with an adiabatic approximation, but it is interesting to consider approximation methods for the relativistic region.

In section 5, we discuss the higher fuzzy sphere case [11–18]. We give a general formula for  $S\text{Tr}(X_i X_i)^m$ , in general irreducible, representations of  $SO(2k+1)$ . This formula is motivated by some considerations surrounding  $D$ -brane charges and the ADHM construction, which are discussed in more detail in [32]. Some of the motivation is explained in appendix A. This allows us a partial discussion of finite  $N$  effects for higher fuzzy spheres. We are able to calculate the physical radius following the argument of section 2; however, in general one needs other symmetrised traces involving elements of the Lie algebra  $so(2k+1)$ .

The symmetrised trace prescription, which we study in detail in this paper, is known to correctly match open string calculations up to the first two orders in an  $\alpha'$  expansion, but the correct answer deviates from the  $(\alpha')^3$  term onwards [19–22]. It is possible however that for certain special symmetric background configurations, it may give the correct physics to all orders. The  $D$ -brane charge computation discussed in the appendix can be viewed as a

possible indication in this direction. In any case, it is important to study the corrections coming from this prescription to all orders, in order to be able to systematically modify it, if that becomes necessary when the correct non-Abelian  $D$ -brane action is known. Conversely the physics of collapsing  $D$ -branes can be used to constrain the form of the non-Abelian DBI.

## 2. Lorentz invariance and the physical radius

We will study the collapse of a cluster of  $N$   $D0$ -branes in the shape of a fuzzy  $S^{2k}$ , in a flat background. This configuration is known to have a large- $N$  dual description in terms of spherical  $D(2k)$  branes with  $N$  units of flux. The microscopic  $D0$  description can be obtained from the non-Abelian action for a number of coincident branes, proposed in [1, 2]

$$S_0 = -\frac{1}{g_s \ell_s} \int dt \text{STr} \sqrt{-\det(M)}, \tag{2.1}$$

where

$$M = \begin{pmatrix} -1 & \lambda \partial_t \Phi_j \\ -\lambda \partial_t \Phi_i & Q_{ij} \end{pmatrix}. \tag{2.2}$$

Here  $a, b$  are worldvolume indices, the  $\Phi$ 's are worldvolume scalars,  $\lambda = 2\pi \ell_s^2$  and

$$Q_{ij} = \delta_{ij} + i\lambda [\Phi_i, \Phi_j]. \tag{2.3}$$

We will consider the time dependent ansatz

$$\Phi_i = \hat{R}(t) X_i, \tag{2.4}$$

The  $X_i$  are matrices obeying some algebra. The part of the action that depends purely on the time derivatives and survives when  $\hat{R} = 0$  is

$$S_{D0} = \int dt \text{STr} \sqrt{1 - \lambda^2 (\partial_t \Phi_i)^2} = \int dt \text{STr} \sqrt{1 - \lambda^2 (\partial_t \hat{R})^2 X_i X_i}. \tag{2.5}$$

For the fuzzy  $S^2$ , the  $X_i = \alpha_i$ , for  $i = 1, 2, 3$ , are generators of the irreducible spin  $n/2$  matrix representation of  $su(2)$ , with matrices of size  $N = n + 1$ . In this case the algebra is

$$[\alpha_i, \alpha_j] = 2i\epsilon_{ijk} \alpha_k \tag{2.6}$$

and following [5], the action for  $N$   $D0$ -branes can be reduced to

$$S_0 = -\frac{1}{g_s \ell_s} \int dt \text{STr} \sqrt{1 + 4\lambda^2 \hat{R}^4 \alpha_i \alpha_i} \sqrt{1 - \lambda^2 (\partial_t \hat{R})^2 \alpha_i \alpha_i}. \tag{2.7}$$

If we define the physical radius using

$$R_{\text{phys}}^2 = \lambda^2 \lim_{m \rightarrow \infty} \frac{\text{STr}(\Phi_i \Phi_i)^{m+1}}{\text{STr}(\Phi_i \Phi_i)^m} = \lambda^2 \hat{R}^2 \lim_{m \rightarrow \infty} \frac{\text{STr}(\alpha_i \alpha_i)^{m+1}}{\text{STr}(\alpha_i \alpha_i)^m}, \tag{2.8}$$

we will find that the Lagrangian will be convergent for speeds between 0 and 1. The radius of convergence will be exactly one — this follows by applying the ratio test to the series expansion of

$$STr\sqrt{1 - \lambda^2 \dot{\hat{R}}^2 \alpha_i \alpha_i}, \tag{2.9}$$

where a dot indicates differentiation with respect to time. This leads to

$$R_{\text{phys}}^2 = \lambda^2 \hat{R}^2 n^2. \tag{2.10}$$

Using explicit formulae for the symmetrised traces we will also see that, with this definition of the physical radius, the formulae for the Lagrangian and energy will have a first singularity at  $\dot{R}_{\text{phys}} = 1$ . In the large  $n$  limit, the definition of physical radius in (2.10) agrees with [2], where  $R_{\text{phys}}$  is defined by  $R_{\text{phys}}^2 = \frac{\lambda^2}{N} Tr(\Phi_i \Phi_i)$ . Note that this definition of the physical radius will also be valid for the higher dimensional fuzzy spheres, and more generally in any matrix construction, where the terms in the non-Abelian DBI action depending purely on the velocity, are of the form  $\sqrt{1 - \lambda^2 X_i X_i (\partial_t \hat{R})^2}$ .

In what follows, the sums we get in expanding the square root are conveniently written in terms of  $r, s$ , defined by  $r^4 = 4\lambda^2 \hat{R}^4$  and  $s^2 = \lambda^2 \dot{\hat{R}}^2$ . It is also useful to define

$$\begin{aligned} L^2 &= \frac{\lambda n}{2}, \\ \hat{r}^2 &= \frac{R_{\text{phys}}^2}{L^2} = r^2 n, \\ \hat{s}^2 &= s^2 n^2. \end{aligned} \tag{2.11}$$

The  $\hat{r}$  and  $\hat{s}$  variables approach the variables called  $r, s$  in the large- $n$  discussion of [5, 23]. Note, using (2.10), that

$$\dot{R}_{\text{phys}}^2 = s^2 n^2 = \hat{s}^2. \tag{2.12}$$

### 3. The fuzzy $S^2$ at finite $n$

For the fuzzy  $S^2$ , the relevant algebra is that of  $su(2)$ , equation (2.6) above. We also have the Casimir

$$c = \alpha_i \alpha_i = (N^2 - 1),$$

where the last expression gives the value of the Casimir in the  $N$ -dimensional representation where  $N = n + 1$ , and  $n$  is related to the spin  $J$  by  $n = 2J$ .

We present here the result of the full evaluation of the symmetrised trace for odd  $n$

$$C(m, n) \equiv \frac{1}{n+1} STr(\alpha_i \alpha_i)^m = \frac{2(2m+1)}{n+1} \sum_{i=1}^{(n+1)/2} (2i-1)^m, \tag{3.1}$$

whilst for even  $n$

$$C(m, n) \equiv \frac{1}{n+1} STr(\alpha_i \alpha_i)^m = \frac{2(2m+1)}{n+1} \sum_{i=1}^{n/2} (2i)^m. \tag{3.2}$$

For  $m = 0$  the second expression doesn't have a correct analytic continuation and we will impose the value  $STr(\alpha_i\alpha_i)^0 = 1$ . The expression for  $C(m, 1)$  was proved in [5]. A proof of (3.2) for  $n = 2$  is given in appendix B. The general formulae given above are conjectured on the basis of various examples, together with arguments related to  $D$ -brane charges. These are given in appendix A. There is also a generalisation to the case of higher dimensional fuzzy spheres, described in section 6 and the appendices.

We will now use the results (3.1), (3.2), to obtain the symmetrised trace corrected energy for a configuration of  $N$  time dependent  $D0$ -branes blown up to a fuzzy  $S^2$ . The reduced action (2.7) can be expanded to give

$$\begin{aligned} \mathcal{L} &= -STr\sqrt{1 + 4\lambda^2\hat{R}^4\alpha_i\alpha_i}\sqrt{1 - \lambda^2\dot{\hat{R}}^2\alpha_i\alpha_i} \\ &= -STr\sqrt{1 + r^4\alpha_i\alpha_i}\sqrt{1 - s^2\alpha_i\alpha_i} \end{aligned} \tag{3.3}$$

$$= -STr \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} s^{2m} r^{4l} (\alpha_i\alpha_i)^{m+l} \binom{1/2}{m} \binom{1/2}{l} (-1)^m. \tag{3.4}$$

The expression for the energy then follows directly -

$$\mathcal{E} = -STr \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} s^{2m} r^{4l} (2m - 1) (\alpha_i\alpha_i)^{m+l} \binom{1/2}{m} \binom{1/2}{l} (-1)^m, \tag{3.5}$$

and after applying the symmetrised trace results given above we get the finite- $n$  corrected energy for any finite-dimensional irreducible representation of spin- $\frac{n}{2}$  for the fuzzy  $S^2$ .

For  $n = 1, 2$  one finds

$$\frac{1}{2}\mathcal{E}_{n=1}(r, s) = \frac{1 + 2r^4 - r^4s^2}{\sqrt{1 + r^4(1 - s^2)^{3/2}}}, \tag{3.6}$$

$$\frac{1}{3}\mathcal{E}_{n=2}(r, s) = \frac{2}{3} \frac{(1 + 8r^4 - 16r^4s^2)}{\sqrt{1 + 4r^4(1 - 4s^2)^{3/2}}} + \frac{1}{3}. \tag{3.7}$$

We note that both of these expressions provide equations of motion which are solvable by solutions of the form  $\hat{r} = t$ .

For the case of general  $n$ , it can be checked that the energy can be written

$$\mathcal{E}_n(r, s) = \sum_{l=1}^{\frac{n+1}{2}} \frac{2 - 2(2l - 1)^2r^4((2l - 1)^2s^2 - 2)}{\sqrt{1 + (2l - 1)^2r^4(1 - (2l - 1)^2s^2)^{3/2}}}, \tag{3.8}$$

for  $n$ -odd, while for  $n$ -even

$$\mathcal{E}_n(r, s) = 1 + \sum_{l=1}^{\frac{n}{2}} \frac{2 - 2(2l)^2r^4((2l)^2s^2 - 2)}{\sqrt{1 + (2l)^2r^4(1 - (2l)^2s^2)^{3/2}}}. \tag{3.9}$$

Equivalently, the closed form expression for the Lagrangian for  $n$ -odd is

$$\mathcal{L}_n(r, s) = -2 \sum_{l=1}^{\frac{n+1}{2}} \frac{1 - 2(2l - 1)^2s^2 + (2l - 1)^2r^4(2 - 3(2l - 1)^2s^2)}{\sqrt{1 + (2l - 1)^2r^4}\sqrt{1 - (2l - 1)^2s^2}}, \tag{3.10}$$

whilst for  $n$ -even

$$\mathcal{L}_n(r, s) = -1 - 2 \sum_{l=1}^{\frac{n}{2}} \frac{1 - 2(2l)^2 s^2 + (2l)^2 r^4 (2 - 3(2l)^2 s^2)}{\sqrt{1 + (2l)^2 r^4} \sqrt{1 - (2l)^2 s^2}}. \quad (3.11)$$

It is clear from these expressions that the equations of motion in the higher spin case will also admit the  $\hat{r} = t$  solution. Note that, after performing the rescaling to physical variables (2.11) and (2.12), these energy functions and Lagrangians have no singularity for fixed  $r$ , in the region  $0 \leq \hat{R}_{\text{phys}} \leq 1$ . As  $s$  increases from 0 the first singularity is at  $s = \frac{1}{n}$ , which corresponds to  $\hat{s} = \hat{R}_{\text{phys}} = 1$ . In this sense they are consistent with a fixed speed of light. However, they do not involve, for fixed  $r$ , the form  $\sqrt{dt^2 - dr^2}$  and hence do not have an  $so(1, 1)$  symmetry. It will be interesting to see if there are generalisations of  $so(1, 1)$ , possibly involving non-linear transformations of  $dt, dr$ , which can be viewed as symmetries.

### 3.1 The $D1 \perp D3$ intersection at finite- $n$

The static  $D1 \perp D3$  system consists of a set of  $N$   $D$ -strings ending on an orthogonal  $D3$  [7, 8]. Far away from the intersection, the valid description is in terms of the non-abelian  $D1$ -brane worldvolume action, describing a funnel of increasing fuzzy- $S^2$  cross-section. The abelian  $D3$  worldvolume picture is valid close to the intersection, describing a BPS magnetic monopole with its Higgs field interpreted as an infinite spike transverse to the brane. In the large- $N$  limit the two descriptions overlap significantly.

The relationship between the microscopic descriptions of the time dependent  $D0 - D2$  system and the static  $D1 \perp D3$  intersection was established in [23]. In that paper, the large- $n$  behaviour of both systems was described by a genus one Riemann surface, which is a fixed orbit in complexified phase space. This was done by considering the conserved energy and pressure and complexifying the variables  $r$  and  $\partial r = s$  respectively. Conservation of the energy-momentum tensor then yielded elliptic curves in  $r, s$ , involving a fixed parameter  $r_0$ , which corresponded to the initial radius of the configuration. The actions for the two systems were related by a Wick rotation.

We can apply the symmetrised trace formula to also get exact results for the corrected pressure of the fuzzy- $S^2$  funnel configuration at finite- $n$ . For our system we simply display the general result and the first two explicit cases

$$\mathcal{P} = STr \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} s^{2m} r^{4l} (2m - 1) (\alpha_i \alpha_i)^{m+l} \binom{1/2}{m} \binom{1/2}{l} \quad (3.12)$$

$$\frac{1}{2} \mathcal{P}_{n=1}(r, s) = -\frac{1 + 2r^4 + r^4 s^2}{\sqrt{1 + r^4} (1 + s^2)^{3/2}} \quad (3.13)$$

$$\frac{1}{3} \mathcal{P}_{n=2}(r, s) = -\frac{2(1 + 8r^4 + 16r^4 s^2)}{3\sqrt{1 + 4r^4} (1 + 4s^2)^{3/2}} - \frac{1}{3}. \quad (3.14)$$

Note that the above formulae are related to the energies in (3.6) and (3.7) by a Wick rotation  $s \rightarrow is$ . Similar results to those for the time dependent case hold for the exact expression of the pressure for the general spin- $\frac{n}{2}$  representation. Note again that these expressions will provide equations of motion which are solved by solutions of the form

$\hat{r} = 1/\sigma$ , where  $\sigma$  is the spatial  $D1$  worldvolume coordinate. An easy way to see this is to substitute  $s^2 = r^4$  in (3.13), (3.14), to find that the pressures becomes independent of  $r$  and  $s$ . Since the higher spin results for the pressure are sums of the  $n = 1$  or  $n = 2$  cases, the argument extends.

### 3.2 Finite $N$ dynamics as a quotient of free multi-particle dynamics

Using the formulae above, we can see that the fuzzy  $S^2$  energy for general  $n$  is determined by the energy at  $n = 1$ . In the odd  $n$  case

$$C(m, n) = \frac{2}{n+1} C(m, 1) \sum_{i_3=1}^{\frac{n+1}{2}} (2i_3 - 1)^{2m} = \frac{2}{n+1} (2m+1) \sum_{i_3=1}^{\frac{n+1}{2}} (2i_3 - 1)^{2m} .$$

Using this form for  $C(m, n)$  in the derivation of the energy, we get

$$\mathcal{E}_n(r, s) = \sum_{i_3=1}^{\frac{n+1}{2}} \mathcal{E}_{n=1} \left( r\sqrt{(2i_3 - 1)}, s(2i_3 - 1) \right) . \tag{3.15}$$

Similarly, in the even  $n$  case, we find

$$\mathcal{E}_n(r, s) = \sum_{i_3=1}^{\frac{n}{2}} \mathcal{E}_{n=2}(r\sqrt{i_3}, s(i_3)) . \tag{3.16}$$

It is also possible to write  $C(m, 2)$  in terms of  $C(m, 1)$  as (for  $m \neq 0$ )

$$C(m, 2) = \frac{2^{2m+1}}{3} C(m, 1) = \frac{2^{2m+1}}{3} (2m+1) . \tag{3.17}$$

Thus we can write  $\mathcal{E}_n(r, s)$ , for even  $n$ , in terms of the basic  $\mathcal{E}_{n=1}(r, s)$  as

$$\mathcal{E}_n(r, s) = 1 + \sum_{i_3=1}^{\frac{n}{2}} \mathcal{E}_{n=1}(r\sqrt{(2i_3)}, s(2i_3)) . \tag{3.18}$$

These expressions for the energy of spin  $n/2$  can be viewed as giving the energy in terms of a quotient of a multi-particle system, where the individual particles are associated with the spin half system. For example, the energy function for  $(n+1)/2$  free particles with dispersion relation determined by  $\mathcal{E}_{n=1}$  is  $\sum_i \mathcal{E}_{n=1}(r_i, s_i)$ . By constraining the particles by  $r_i = r\sqrt{2i+1}$ ,  $s_i = s(2i+1)$  we recover precisely (3.15).

We can now use this result to resolve a question raised by [5] on the exotic bounces seen in the Lagrangians obtained by keeping a finite number of terms in the  $1/n$  expansion. With the first  $1/n$  correction kept, the bounce appeared for a class of paths involving high velocities with  $\gamma = \frac{1}{\sqrt{1-s^2}} \sim c^{1/4}$ , near the limit of validity of the  $1/n$  expansion. The bounce disappeared when two orders in the expansion were kept. It was clear that whether the bounces actually happened or not could only be determined by finite  $n$  calculations. These exotic bounces would be apparent in constant energy contour plots for  $r, s$  as a zero in the first derivative  $\partial r/\partial s$ . In terms of the energies, this translates into the presence of



a zero of  $\partial E/\partial s$  for constant  $r$ . It is easy to show from the explicit forms of the energies that these quantities are strictly positive for  $n = 1$  and  $n = 2$ . Since the energy for every  $n$  can be written in terms of these, we conclude that there are no bounces for any finite  $n$ . This resolves the question raised in [5] about the fate at finite  $n$  of these bounces.

We note that the large- $n$  limit of the formula for the energy provides us with a consistency check. In the large  $n$ -limit the sums above become integrals. For the odd- $n$  case (even- $n$  can be treated in a similar fashion), define  $x = \frac{2i_3-1}{n} \sim \frac{2i_3}{n}$ . Then the sum in (3.15) goes over to the integral

$$\frac{n}{2} \int_0^1 dx \frac{2 - 2x^2 n^2 r^4 (x^2 n^2 s^2 - 2)}{\sqrt{1 + x^2 n^2 r^4 (1 - x^2 s^2)^{3/2}}} = \frac{n\sqrt{1 + r^4 n^2}}{\sqrt{1 - s^2 n^2}}. \quad (3.19)$$

By switching to the  $\hat{r}, \hat{s}$  parameters the energy can be written as  $\frac{n\sqrt{1+\hat{r}^4}}{\sqrt{1-\hat{s}^2}}$ . This matches exactly the large  $n$  limit used in [5].

## 4. Physical properties of the finite $N$ solutions

### 4.1 Special limits where finite $n$ and large $n$ formulae agree

In the above we compared the finite  $n$  formula with the large  $n$  limit. Here we consider the comparison between the fixed  $n$  formula and the large  $n$  one in some other limits. On physical grounds we expect some agreement. The  $D0 - D2$  system at large  $\hat{r}$  and small velocity  $\hat{s}$  is expected to be correctly described by the  $D2$  equations. These coincide with the large  $n$  limit of the  $D0$ . In the  $D1 \perp D3$  system, the large  $\hat{r}$  limit with large imaginary  $\hat{s}$  is also described by the  $D3$ .

Such an argument should extend to the finite- $n$  case. In [23], these systems were simply described by a genus one Riemann surface. However, in this case the energy functions are more complicated and the resulting Riemann surfaces are of higher genus. We still expect the region of the finite  $n$  curve, with large  $r$  and small, real  $s$ , to agree with the same limit of the large  $n$  curve. We also expect the region of large  $r$  and large imaginary  $s$  to agree with large  $n$ .

For concreteness consider odd  $n$ . Indeed for large  $r$ , small  $s$ , (3.8) gives

$$\sum_{l=1}^{\frac{n+1}{2}} 4(2l-1)r^2 \sim n^2 r^2 = n\hat{r}^2 \quad (4.1)$$

This agrees with the result obtained from the  $D2$ -brane Lagrangian [5] using (2.11) and (2.12). In this limit, both the genus one curve and the high genus finite  $n$  curves degenerate to a pair of points. Now consider the energy functions in the limit of large  $\hat{r}$  and large imaginary  $\hat{s}$ . This is the right regime for comparison with the  $D1 \perp D3$  system since this is described, at large  $n$  and in the region of large  $\hat{r}$ , by  $\hat{r} \sim \frac{1}{\sigma}$ . This means that  $\hat{r}$  is large at small  $\sigma$ , where  $\frac{d\hat{r}}{d\sigma} \rightarrow i\hat{s}$  is large. Using the Wick rotation  $s \rightarrow is$  (which takes us from the time-dependent system to the space-dependent system)

$$\mathcal{E} \rightarrow \mathcal{P} \sim \frac{nr^2}{s} = n\frac{\hat{r}^2}{\hat{s}}, \quad (4.2)$$

which agrees with the same limit of the large  $n$  curve. In this limit, both the large  $n$  genus one curve and the finite  $n$  curves of large genus degenerate to a genus zero curve.

The agreement in (4.1) between the  $D0$  and  $D2$  pictures is a stringy phenomenon. It follows from the fact that there is really one system, a bound state of  $D0$  and  $D2$  branes. A boundary conformal field theory would have boundary conditions that encode the presence of both the  $D0$  and  $D2$ . In the large  $N$  limit, the equations of motion coming from the  $D0$ -effective action agree with the  $D2$ -effective action description at all  $R_{\text{phys}}$ . This is because at large  $N$  it is possible to specify a DBI-scaling where the regime of validity of both the  $D0$  and  $D2$  effective actions extends for all  $R_{\text{phys}}$ . This follows because the DBI scaling has  $\ell_s \rightarrow 0$  [24]. Indeed it is easy to see that the effective open string metric discussed in [24] has the property that  $\ell_s^2 G^{-1} = \frac{\ell_s^2 R_{\text{phys}}^2}{R_{\text{phys}}^4 + L^4}$  goes to zero when  $N \rightarrow \infty$  with  $L = \ell_s \sqrt{\pi N}$ ,  $R_{\text{phys}}$  fixed. This factor  $\ell_s^2 G^{-1}$  controls higher derivative corrections for the open string degrees of freedom. At finite  $N$ , we can keep  $\ell_s^2 G^{-1}$  small, either when  $R_{\text{phys}} \ll L$  or  $R_{\text{phys}} \gg L$ . Therefore, there are two regimes where the stringy description reduces to an effective field theory, where higher derivatives can be neglected. The agreement holds for specified regions of  $R_{\text{phys}}$  as well as  $\dot{R}_{\text{phys}}$ , because the requirement  $\ell_s^2 G^{-1} \ll 1$  is not the only condition needed to ensure that higher derivatives can be neglected. We also require that the proper acceleration is small. At large  $R_{\text{phys}}$ , the magnetic flux density is small (as well as the higher derivatives being small) and the  $D2$ -brane without non-commutativity is a good description. This is why the finite  $N$  equations derived from the  $D0$ -brane effective field theory agree with the Abelian  $D2$ -picture. For small  $R_{\text{phys}}$ , small  $\dot{R}_{\text{phys}}$ , we can also neglect higher derivatives. This is the region where the  $D0$ -Yang-Mills description is valid, or equivalently a strongly non-commutative  $D2$ -picture.

**4.2 Finite  $N$  effects: time of collapse, proper accelerations and violation of the dominant energy condition**

We will consider the time of collapse as a function of  $n$  using the definition of the physical radius given in section 2. In order to facilitate comparison with the large  $n$  system, we will be using  $\hat{r}, \hat{s}$  variables. To begin with, consider the dimensionless acceleration, which can be expressed as

$$-\hat{s} \frac{\partial_{\hat{s}} \mathcal{E}|_{\hat{r}}}{\partial_{\hat{r}} \mathcal{E}|_{\hat{s}}}, \tag{4.3}$$

with  $\gamma = 1/\sqrt{(1 - \hat{s}^2)}$ . As the sphere starts collapsing from  $\hat{r} = \hat{r}_0$  down to  $\hat{r} = 0$ , the speed changes from  $\hat{s} = 0$  to a value less than  $\hat{s} = 1$ . It is easy to see that the acceleration does not change sign in this region. Using the basic energy  $\hat{\mathcal{E}} = \mathcal{E}/N$  from (3.6), we can write

$$\begin{aligned} \frac{\partial \hat{\mathcal{E}}_{n=1}(\hat{r}, \hat{s})}{\partial \hat{s}} &= \hat{s} \frac{(3(1 + \hat{r}^4) + \hat{r}^4(1 - \hat{s}^2))}{\sqrt{(1 + \hat{r}^4)(1 - \hat{s}^2)}^{\frac{5}{2}}}, \\ \frac{\partial \hat{\mathcal{E}}_{n=1}(\hat{r}, \hat{s})}{\partial \hat{r}} &= \frac{2\hat{r}^3}{(1 + \hat{r}^4)^{\frac{3}{2}}(1 - \hat{s}^2)^{\frac{3}{2}}} \left( (1 + \hat{r}^4) + (1 - \hat{s}^2)(2 + \hat{r}^4) \right). \end{aligned} \tag{4.4}$$

Neither of the partial derivatives change sign in the range  $\hat{s} = 0$  to 1. Hence the speed  $\hat{s}$  increases monotonically. The same result is true for  $n > 1$ , since the energy functions for all these cases can be written as a sum of the energies at  $n = 1$ .

In the  $n = 1$  case,  $\hat{r} = r$ ,  $\hat{s} = s$ . For fixed  $r_0$  the speed at  $r = 0$  is given by

$$(1 - s^2|_{n=1}) = \frac{(1 + r_0^4)^{\frac{1}{3}}}{(1 + 2r_0^4)^{\frac{2}{3}}}. \tag{4.5}$$

Comparing this with the large  $n$  formula, which follows from conservation of energy (see for example section 2 of [5])

$$(1 - s^2|_{n=\infty}) = (1 + r_0^4)^{-1}, \tag{4.6}$$

it is easy to see that

$$\left( \frac{(1 - s^2)|_{n=\infty}}{(1 - s^2)|_{n=1}} \right)^3 = \frac{(1 + 2r_0^4)^2}{(1 + r_0^4)^4} < 1, \tag{4.7}$$

which establishes that the speed at  $r = 0$  is larger for  $n = \infty$ .

We can strengthen this result to show that the speed of collapse at all  $r < r_0$  is smaller for  $n = 1$  than at  $n = \infty$ . For any  $r < r_0$  we evaluate this energy function with the speed of collapse evaluated at  $s^2 = \frac{r_0^4 - r^4}{r_0^4 + 1}$ , which is the speed at the same  $r$  in the large  $n$  problem. Let us define  $F(r, r_0) = \hat{\mathcal{E}}_{n=1} \left( r, s = \sqrt{\frac{r_0^4 - r^4}{r_0^4 + 1}} \right)$ . We compare this with  $\hat{\mathcal{E}}_{n=\infty}(r, s)$  for  $s$  appropriate for the  $n = 1$  problem, which is just  $\frac{1 + 2r_0^4}{\sqrt{1 + r_0^4}} \equiv G(r_0)$  by conservation of energy. We now use the fact, established above, that  $\frac{\partial \hat{\mathcal{E}}_{n=1}}{\partial s}$  is positive for any real  $r$ . This means that we can show  $s|_{n=1} < \sqrt{\frac{r_0^4 - r^4}{r_0^4 + 1}}$  by showing that  $F(r, r_0) > G(r_0)$ . A short calculation gives

$$F(r, r_0) - G(r_0) = \frac{r_0^4}{\sqrt{1 + r_0^4}(1 + r^4)}(r_0^4 - r^4). \tag{4.8}$$

It is clear that we have the desired inequality, showing that, at each  $r$ , the speed  $s$  in the  $n = 1$  problem is smaller than the speed in the  $n = \infty$  system. Hence the time of collapse is larger at  $n = 1$ . In the  $n = 2$  case, we find that an exactly equivalent treatment proves again that the collapse is slower than at large  $n$ . However, this trend is not a general feature for all  $n$ . In the leading large- $N$  limit, the time of collapse is given by the formula

$$\frac{T}{L} = \int dr \frac{\sqrt{1 + r_0^4}}{\sqrt{r_0^4 - r^4}} = \frac{K(\frac{1}{\sqrt{2}}) \sqrt{R^4 + L^4}}{\sqrt{2} R} \tag{4.9}$$

For fixed  $\ell_s$ ,  $L$  decreases with decreasing  $N$  and as a result  $T$  decreases. When we include the first  $1/N$  correction the time of collapse is [5]

$$\frac{T}{L} = \int dr \left[ \frac{\sqrt{1 + r_0^4}}{\sqrt{r_0^4 - r^4}} + \frac{r_0^8}{6N^2(1 + r_0^4)^{3/2} \sqrt{r_0^4 - r^4}} - \frac{r_0^4(1 + 3(1 + r_0^4))}{6N^2(1 + r_0^4) \sqrt{1 + r_0^4} \sqrt{r_0^4 - r^4}} \right]. \tag{4.10}$$

By performing numerical integration of the above for several values of the parameter  $r_0$  and some large but finite values of  $N$ , we see that the time of collapse is smaller for the  $1/N$  corrected case. This means that, in the region of large  $N$  the time of collapse decreases as  $N$  decreases, with both the leading large  $N$  formula and the  $1/N$  correction being consistent with this trend. However, as we saw above the time of collapse at  $n = 1$  and  $n = 2$  are larger than at  $n = \infty$ . This means that there are one or more turning points in the time of collapse as a function of  $n$ .

The deceleration effect that arises in the comparison of  $n = 1$  and  $n = 2$  with large  $n$  may have applications in cosmology. Deceleration mechanisms coming from DBI actions have been studied in the context of bulk causality in AdS/CFT [25, 26] and applied in the problem of satisfying slow roll conditions in stringy inflation [27]. Here we see that the finite  $n$  effects result in a further deceleration in the region of small  $n$ .

We turn to the proper acceleration which is important in checking the validity of our action. Since the DBI action is valid when higher derivatives are small, it is natural to demand that the proper acceleration, should be small (see for example [25]). The condition is  $\gamma^3 \ell_s \partial_t^2 R_{\text{phys}} \ll 1$ . In terms of the dimensionless variables it is  $\gamma^3 (\partial_\tau^2 \hat{r}) \ll \sqrt{N}$ . If we want a trajectory with initial radius  $\hat{r}_0$  such that the proper acceleration always remains less than one through the collapse, then there is an upper bound on  $\hat{r}_0$  (see for example section 8 of [5]). The corresponding upper bound on the physical radius goes to infinity as  $N \rightarrow \infty$ , since  $R_{\text{phys}} \sim \hat{r} \sqrt{\lambda N}$ . For small  $\hat{r}_0$  we are in the Matrix Theory limit and the effective action is valid. For large  $\hat{r}_0$  and  $\hat{r}$ -large, the acceleration is under control,  $\alpha \sim 1/\hat{r}$  and the velocity will be close to zero. Interestingly, there will also be a class of trajectories parametrised by large  $\hat{r}_0$ , which admit relativistic motion. Consider for example the  $n = 1$  case (where  $\hat{r} = r, \hat{s} = s$ ). The proper acceleration can be written as

$$\alpha = -\frac{2r^3}{1+r^4} \frac{-3+2s^2+r^4(s^2-2)}{\sqrt{1-s^2}(r^4(s^2-4)-3)}. \tag{4.11}$$

For  $s \sim 1$  and small  $r$ , this becomes

$$\alpha \simeq -\frac{2r^3}{3\sqrt{1-s^2}} \tag{4.12}$$

and  $\sqrt{1-s^2}$  can be found from the energy at the same limits, in which (3.6) becomes

$$\sqrt{1-s^2} \simeq \frac{1}{(2r_0^2)^{1/3}}. \tag{4.13}$$

Therefore, we can identify a region where the proper acceleration is small by restricting it to be of order  $1/r_0$  for example

$$\alpha \simeq \frac{2r^3}{3} (2r_0^2)^{1/3} \sim \frac{1}{r_0}. \tag{4.14}$$

This means that in regions where  $r \sim r_0^{-5/9}$ , we will have a relativistic limit described by the DBI, where stringy corrections can be neglected. This result also holds in the large- $N$

limit. It will be interesting to develop a perturbative approximation which systematically includes stringy effects away from this region.

Another quantity of interest is the effective mass squared  $E^2 - p^2$ , where  $p = \partial\mathcal{L}/\partial s$  is the radial conjugate momentum. It becomes negative for sufficiently large velocities. This includes the above regime of relativistic speeds and small radii. It is straightforward to see that if our collapsing configuration is considered as a source for spacetime gravity, this implies a violation of the dominant energy condition. In the context of the BFSS matrix model, it has been shown that for an action containing a background spacetime  $G_{IJ} = \eta_{IJ} + h_{IJ}$ , in the linearised approximation, linear couplings in the fluctuation  $h_{0I}$  correspond to momentum in the  $X^I$  direction [28]. The same argument can be developed here for the non-Abelian DBI. We couple a small fluctuation  $h_{0r}$ , which in classical geometry we can write as  $h_{0i} = h_{0r}x_i$  for the unit sphere. We replace  $x_i$  by  $\alpha_i/n$ . The action for D0-branes [1, 2] is generalised from (2.1) by replacing  $\dot{R}$  in  $\lambda\partial_t\Phi_i = \lambda(\dot{R})\alpha_i = \frac{\dot{R}}{n}\alpha_i$  with  $(\dot{R} + h_{0r})$ . It is then clear that the variation with respect to  $\dot{R}$ , which gives  $p$ , is the same as the variation with respect to  $h_{0r}$ , which gives  $T^{0r}$ . Hence, the dominant energy condition will be violated, since  $E < |p|$  is equivalent to  $T^{00} < T^{0r}$ . The violation of this condition by stringy  $D$ -brane matter can have profound consequences. For a discussion of possible consequences in cosmology see [29]. In this context, it is noteworthy that the violation can occur near a region of zero radius, which could be relevant to a near-big-bang region in a braneworld scenario.

### 4.3 Distance to blow-up in $D1 \perp D3$

Comparisons between the finite and large  $N$  results can be made in the spatial case using the conserved pressure. The arguments are similar to what we used for the time of collapse using the energy functions. Consider the case  $n = 1$ , and let  $\hat{P} = P/N$ . First calculate the derivative of the pressure -

$$\frac{\partial\hat{P}}{\partial s} = \frac{s(4r^4 + r^4s^2 + 3)}{\sqrt{1 + r^4(1 - s^2)^{5/2}}}. \tag{4.15}$$

This is clearly always positive. Now evaluate

$$\hat{P}\left(r, s = \frac{\sqrt{r^4 - r_0^4}}{\sqrt{1 + r_0^4}}\right) = -\frac{(1 + r_0^4)^{1/2}}{1 + r^4}(1 + r_0^4 + r^4). \tag{4.16}$$

This should be compared with  $\hat{P}(r, s)$ , evaluated for the value of  $s$  which solves the  $n = 1$  equation of motion, which by conservation of pressure is  $-\frac{(1+2r_0^4)}{\sqrt{1+r_0^4}}$ .

Take the difference to find

$$\hat{P}\left(r, s = \frac{\sqrt{r^4 - r_0^4}}{\sqrt{1 + r_0^4}}\right) + \frac{(1 + 2r_0^4)}{\sqrt{1 + r_0^4}} = \frac{r_0^4(r^4 - r_0^4)}{\sqrt{1 + r_0^4}(1 + r^4)}. \tag{4.17}$$

Thus at fixed  $r_0$  and  $r$ ,  $\hat{P}_{n=1}$ , when evaluated for the value of  $s$  which solves the large  $n$  problem, is larger than when it is evaluated for the value of  $s$  which solves the  $n = \infty$  problem. Since  $\hat{P}$  increases monotonically with  $s$  for fixed  $r$ , this shows that for fixed  $r_0$ ,

and any  $r, s$  is always larger in the large  $N$  problem. Since  $\Sigma = \int dr/s$ , this means the distance to blow-up is smaller for  $n = \infty$ . Hence for fixed  $r_0$ , the distance to blow-up is larger at  $n = 1$ .

### 5. Towards a generalisation to higher even-dimensional fuzzy-spheres

For generalisations to higher dimensional brane systems, and to higher dimensional fuzzy spheres [13, 15–17], it is of interest to derive an extension of the expressions for the symmetrised traces given above. In the general case, we define  $N(k, n)$  to be the dimension of the irreducible representation of  $SO(2k+1)$  with Dynkin label  $(\frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2})$  which contains  $k$  entries. We then take  $C(m, k, n)$  to be the action of the symmetrised trace on  $m$  pairs of matrices  $X_i$ , where  $i = 1, \dots, 2k+1$

$$C(m, k, n) = \frac{1}{N(k, n)} STr(X_i X_i)^m. \tag{5.1}$$

Finding an expression for  $C(m, k, n)$  is non-trivial. Investigations based upon intuition from the ADHM construction lead us to conjecture that for  $n$  odd

$$C(m, k, n) = \frac{2^k \prod_{i_1=1}^k (2m-1+2i_1)}{(k-1)! \prod_{i_2=1}^{2k-1} (n+i_2)} \sum_{i_3=1}^{\frac{n+1}{2}} \left[ \prod_{i_4=1}^{k-1} \left( \left( \frac{n}{2} + i_4 \right)^2 - \left( i_3 - \frac{1}{2} \right)^2 \right) (2i_3-1)^{2m} \right], \tag{5.2}$$

while for  $n$  even<sup>1</sup>

$$C(m, k, n) = \frac{2^k \prod_{i_1=1}^k (2m-1+2i_1)}{(k-1)! \prod_{i_2=1}^{2k-1} (n+i_2)} \sum_{i_3=1}^{\frac{n}{2}} \left[ \prod_{i_4=1}^{k-1} \left( \left( \frac{n}{2} + i_4 \right)^2 - i_3^2 \right) (2i_3)^{2m} \right]. \tag{5.3}$$

We give the arguments leading to the expressions above in appendix A.

For higher even spheres there will be extra complications at finite- $n$ . Consider the case of the fuzzy  $S^4$  for concreteness. The evaluation of the higher dimensional determinant in the corresponding non-Abelian brane action will give expressions with higher products of  $\partial_t \Phi_i$  and  $\Phi_{ij} \equiv [\Phi_i, \Phi_j]$

$$S = -T_0 \int dt STr \left\{ 1 + \lambda^2 (\partial_t \Phi_i)^2 + 2\lambda^2 \Phi_{ij} \Phi_{ji} + 2\lambda^4 (\Phi_{ij} \Phi_{ji})^2 - 4\lambda^4 \Phi_{ij} \Phi_{jk} \Phi_{kl} \Phi_{li} + \right. \\ \left. + 2\lambda^4 (\partial_t \Phi_i)^2 \Phi_{jk} \Phi_{kj} - 4\lambda^4 \partial_t \Phi_i \Phi_{ij} \Phi_{jk} \partial_t \Phi_k + \frac{\lambda^6}{4} (\epsilon_{ijklm} \partial_t \Phi_i \Phi_{jk} \Phi_{lm})^2 \right\}^{1/2}. \tag{5.4}$$

The ansatz for the transverse scalars will still be

$$\Phi_i = \hat{R}(t) X_i,$$

where now  $i = 1, \dots, 5$  and the  $X^i$ 's are given by the action of  $SO(5)$  gamma matrices on the totally symmetric  $n$ -fold tensor product of the basic spinor. After expanding the

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<sup>1</sup>For  $m = 0$  the value  $STr(X_i X_i)^0 = 1$  is once again imposed.

square root, the symmetrisation procedure should take place over all the  $X_i$ 's and  $[X_i, X_j]$ 's. However, the commutators of commutators  $[[X, X], [X, X]]$  will give a nontrivial contribution, as opposed to what happens in the large- $n$  limit where they are sub-leading and are taken to be zero. Therefore, in order to uncover the full answer for the finite- $n$  fuzzy  $S^4$  it is not enough to just know the result of  $STr(X_i X_i)^m$  — we need to know the full  $STr((X X)^{m_1}([X, X][X, X])^{m_2})$  with all possible contractions among the above. It would be clearly interesting to have the full answer for the fuzzy  $S^4$ . A similar story will apply for the higher even-dimensional fuzzy spheres.

Note, however, that for  $\hat{R} = 0$  in (5.4) all the commutator terms  $\Phi_{ij}$  will vanish, since they scale like  $\hat{R}^2$ . This reduces the symmetrisation procedure to the one involving  $X_i X_i$  and yields only one sum for the energy. The same will hold for any even-dimensional  $S^{2k}$ , resulting in the following general expression

$$\begin{aligned} \mathcal{E}_{n,k}(0, s) &= -STr \sum_{m=0}^{\infty} (-1)^m s^{2m} (2m-1) (X_i X_i)^m \binom{1/2}{m} \\ &= -N(k, n) \sum_{m=0}^{\infty} (-1)^m s^{2m} (2m-1) C(m, k, n) \binom{1/2}{m}. \end{aligned} \tag{5.5}$$

Using (5.2), notice that in the odd  $n$  case

$$C(m, k, n) = \prod_{i_2=1}^{2k-1} \frac{(1+i_2)}{(n+i_2)} C(m, k, 1) \sum_{i_3}^{\frac{n+1}{2}} \frac{f_{\text{odd}}(i_3, k, n)}{f_{\text{odd}}(1, k, 1)} (2i_3-1)^{2m}. \tag{5.6}$$

The factor  $f_{\text{odd}}$  is

$$f_{\text{odd}}(i_3, k, n) = \prod_{i_4=1}^{k-1} \left( \left( \frac{n}{2} + i_4 \right)^2 - \left( i_3 - \frac{1}{2} \right)^2 \right). \tag{5.7}$$

Inserting this form for  $C(m, k, n)$  in terms of  $C(m, k, 1)$  we see that

$$\mathcal{E}_{n,k}(0, s) = N(n, k) \prod_{i_2=1}^{2k-1} \frac{(1+i_2)}{(n+i_2)} \sum_{i_3=1}^{\frac{n+1}{2}} \frac{f_{\text{odd}}(i_3, k, n)}{f_{\text{odd}}(1, k, 1)} \hat{\mathcal{E}}_{n=1,k}(0, s(2i_3-1)) \tag{5.8}$$

Similarly we derive, in the even  $n$  case, that

$$\mathcal{E}_{n,k}(0, s) = N(n, k) \prod_{i_2=1}^{2k-1} \frac{(2+i_2)}{(n+i_2)} \sum_{i_3=1}^{\frac{n}{2}} \frac{f_{\text{even}}(i_3, k, n)}{f_{\text{even}}(1, k, 2)} \hat{\mathcal{E}}_{n=2,k}(0, s(i_3)) \tag{5.9}$$

where

$$f_{\text{even}}(i_3, k, n) = \prod_{i_4=1}^{k-1} \left( \left( \frac{n}{2} + i_4 \right)^2 - i_3^2 \right) \tag{5.10}$$

and  $\hat{\mathcal{E}}$  is the energy density, i.e. the energy divided a factor of  $N(n, k)$ .

It is also possible to write  $C(m, k, 2)$  in terms of  $C(m, k, 1)$

$$\begin{aligned}
 C(m, k, 2) &= 2^{2m} C(m, k, 1) \prod_{i_4=1}^{k-1} \frac{i_4(i_4 + 2)}{i_4(i_4 + 1)} \prod_{i_2=1}^{2k-1} \frac{(i_2 + 1)}{(i_2 + 2)} \\
 &= 2^{2m} C(m, k, 1) \frac{f_{\text{even}}(1, k, 2)}{f_{\text{odd}}(1, k, 1)} \prod_{i_2=1}^{2k-1} \frac{(i_2 + 1)}{(i_2 + 2)}, \tag{5.11}
 \end{aligned}$$

which is valid for all values of  $m \neq 0$ .

It turns out to be possible to give explicit forms for the energy for the  $n = 1$  and  $n = 2$  case. Since the definition of the physical radius in section 2 is also valid for higher dimensional fuzzy spheres, we can express the results in terms of the rescaled variables  $\hat{r}$  and  $\hat{s}$

$$\begin{aligned}
 \hat{\mathcal{E}}_{n=1,k}(0, \hat{s}) &= \frac{1}{(1 - \hat{s}^2)^{\frac{2k+1}{2}}} \\
 \hat{\mathcal{E}}_{n=2,k}(0, \hat{s}) &= \frac{1}{(1 - \hat{s}^2)^{\frac{2k+1}{2}}} \frac{(k + 1)}{(2k + 1)}. \tag{5.12}
 \end{aligned}$$

When plugged into (5.8), (5.9) the above results provide a closed form for the energy at  $\hat{r} = 0$ , for any  $n$  and any  $k$ . A complete study of the time dependent dynamics requires the evaluation of the energy functions for all  $\hat{r}$ , but the relative simplicity of (5.12) suggests that the computation of the required additional symmetrised traces might reveal a tractable extension.

## 6. Summary and outlook

We have given a detailed study of the finite  $N$  effects for the time dependent  $D0 - D2$  fuzzy sphere system and the related  $D1 \perp D3$  funnel. This involved calculating symmetrised traces of  $SO(3)$  generators. The formulae have a surprising simplicity.

The energy function  $E(r, s)$  in the large  $N$  limit looks like a relativistic particle with position dependent mass. This relativistic nature is modified at finite  $N$ . Nevertheless our results are consistent with a fixed relativistic upper speed limit. This is guaranteed by an appropriate definition of the physical radius which relies on the properties of symmetrised traces of large numbers of generators. We showed that the exotic bounces found in the large  $N$  expansion in [5] do not occur. It was previously clear that these exotic bounces happened near the regime where the  $1/N$  expansion was breaking down. The presence or absence of these could only be settled by a finite  $N$  treatment, which we have provided in this paper. We also compared the time of collapse of the finite  $N$  system with that of the large  $N$  system and found a finite  $N$  deceleration effect for the first small values of  $N$ . The modified  $E(r, s)$  relation allows us to define an effective squared mass which depends on both  $r, s$ . For certain regions in  $(r, s)$  space, it can be negative. When the  $D0 - D2$  system is viewed as a source for gravity, a negative sign of this effective mass squared indicates that the brane acts as a gravitational source which violates the dominant energy condition.



We extended some of our discussion to the case of higher even fuzzy spheres with  $SO(2k + 1)$  symmetry. The results for symmetrised traces that we obtain can be used in a proposed calculation of charges in the  $D1 \perp D(2k + 1)$  system. They also provide further illustrations of how the correct definition of physical radius using symmetrised traces of large powers of Lie algebra generators gives consistency with a constant speed of light. A more complete discussion of the finite  $N$  effects for the higher fuzzy spheres could start from these results. Generalisations of the finite  $N$  considerations to fuzzy spheres in more general backgrounds [30] will be interesting to consider, with a view to possible applications in cosmology.

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## A. General formula for the symmetrised trace

As in section 5, we define  $N(k, n)$  to be the dimension of the irreducible representation of  $SO(2k + 1)$  with Dynkin label  $(\frac{n}{2}, \frac{n}{2}, \dots, \frac{n}{2})$ . These are the usual fuzzy sphere representations [13, 14] (for example, for  $k=1$  the  $X_i$  are the elements of the Lie algebra of  $SU(2)$  in the irreducible representation with spin  $\frac{n}{2}$ ). Then

$$N(k, n) = \prod_{1 \leq i < j \leq k} \frac{n + 2k - (i + j) + 1}{2k - (i + j) + 1} \prod_{l=1}^k \frac{n + 2k - 2l + 1}{2k - 2l + 1}. \quad (\text{A.1})$$

The symmetrised trace is defined to be the normalised sum over permutations of the matrices

$$STr(X_{i_1} \dots X_{i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} Tr(X_{\sigma(1)} \dots X_{\sigma(p)}). \quad (\text{A.2})$$

We have given earlier a conjecture for the symmetrised trace of  $m$  powers of the quadratic Casimir  $X_i X_i = c \mathbb{I}_{n \times n}$ , where  $c = n(n + 2k)$ . This is, for all  $m, k$  and  $n$  even

$$\frac{1}{N(k, n)} STr(X_i X_i)^m = \frac{2^k \prod_{i_1=1}^k (2m - 1 + 2i_1)}{(k - 1)! \prod_{i_2=1}^{2k-1} (n + i_2)} \sum_{i_3=1}^{\frac{n}{2}} \left[ \prod_{i_4=1}^{k-1} \left( \left( \frac{n}{2} + i_4 \right)^2 - i_3^2 \right) (2i_3)^{2m} \right], \quad (\text{A.3})$$

where for  $k = 1$  the product over  $i_4 = 1, \dots, k - 1$  is just defined to be equal to 1. Similarly for all  $m, k$  and for  $n$  odd we have proposed that

$$\frac{1}{N(k, n)} \text{STr}(X_i X_i)^m = \frac{2^k \prod_{i_1=1}^k (2m - 1 + 2i_1)}{(k - 1)! \prod_{i_2=1}^{2k-1} (n + i_2)} \times \sum_{i_3=1}^{\frac{n+1}{2}} \left[ \prod_{i_4=1}^{k-1} \left( \left( \frac{n}{2} + i_4 \right)^2 - \left( i_3 - \frac{1}{2} \right)^2 \right) (2i_3 - 1)^{2m} \right]. \quad (\text{A.4})$$

The argument leading to these conjectures follows. Firstly, it is possible to view the fuzzy sphere matrices  $X_i$  as the transverse coordinates of the world-volume theory of a stack of  $D1$ -branes expanding into a stack of  $D(2k + 1)$ -branes [8–10]. There is also a dual realisation of this system in which the  $D1$ -branes appear as a monopole in the world-volume theory of the  $D(2k + 1)$ -branes. The ADHM construction can be used to construct the monopole dual to the fuzzy sphere transverse coordinates. If one takes the  $N(k, n)$ -dimensional fuzzy sphere matrices representing a stack of  $N(k, n)$   $D1$ -branes as ADHM data for a monopole, then one naturally constructs a monopole defined on a stack of  $N(k - 1, n + 1)$   $D(2k + 1)$ -branes. We have also done the calculation which shows that the charge for the monopole just constructed gives precisely  $N(k, n)$ , which provides a consistency check (this calculation extends some results in [31] and will appear in [32]).

It is also possible to calculate the number of  $D(2k + 1)$  branes from the fuzzy sphere ansatz for the transverse coordinates, by looking at the RR coupling on the  $D$ -string worldvolume. In this case one does not get  $N(k - 1, n + 1)$  as one would expect, but instead the quantity

$$N(k - 1, n + 1) \frac{\prod_{i=1}^{2k-1} (n + i)}{c^{k-\frac{1}{2}}}, \quad (\text{A.5})$$

where  $c = n(n + 2k)$ . This number of branes does agree with  $N(k - 1, n + 1)$  for the first two orders in the large  $n$  expansion

$$\text{Number of } D(2k + 1)\text{-branes} = N(k - 1, n + 1) \left( 1 + O\left(\frac{1}{n^2}\right) \right). \quad (\text{A.6})$$

Now consider this RR charge calculation more carefully. First take the  $k = 1$  case. Based on the ADHM construction we expect the number of  $D3$ -branes to be  $N(0, n + 1) = 1$ . However, equation (A.5) suggests that the charge calculation gives for  $k = 1$  the answer

$$\frac{n + 1}{c^{\frac{1}{2}}}. \quad (\text{A.7})$$

Suppose that the numerator in the above is correct, but that the denominator is correct only at large  $n$  and that it receives corrections at lower order to make the number of  $D3$ -branes exactly one. Then these corrections need to satisfy

$$1 = (n + 1)(c^{-\frac{1}{2}} + x_1 c^{-\frac{3}{2}} + x_2 c^{-\frac{5}{2}} + \dots). \quad (\text{A.8})$$

It is easy to show that we need  $x_1 = -\frac{1}{2}$  and  $x_2 = \frac{3}{8}$ , by Taylor expanding and using that  $c = n(n + 2)$  for  $k = 1$ . Therefore, we would like to have a group theoretic justification for

the series

$$c^{-\frac{1}{2}} - \frac{1}{2}c^{-\frac{3}{2}} + \frac{3}{8}c^{-\frac{5}{2}} + \dots . \tag{A.9}$$

There exists a formula for the first three terms in the large  $n$  expansion of the  $k = 1$  symmetrised trace operator [5], namely

$$\frac{1}{N(1, n)}STr(X_i X_i)^m = c^m - \frac{2}{3}m(m-1)c^{m-1} + \frac{2}{45}m(m-1)(m-2)(7m-1)c^{m-2} + \dots . \tag{A.10}$$

Now, if we make the choice  $m = -\frac{1}{2}$  in (A.10) we get precisely (A.9). However, this suggests that if this choice is correct, then we should have an all orders prediction for the action of the symmetrised trace operator. Thus, for  $k=1$  we predict that

$$\frac{1}{N(1, n)}STr(X_i X_i)^m \Big|_{m=-\frac{1}{2}} \simeq \frac{1}{(n+1)} , \tag{A.11}$$

where for future reference we consider the left hand side to be equal to the symmetrised trace in a large- $n$  series expansion, as appeared in [5].

Checking the conjecture (A.11) beyond the first three terms in a straightforward fashion, by techniques similar to those employed in [5], proves difficult. This involves either adding up a large number of chord diagrams, or complicated combinatorics if one uses the highest weight method.

An alternative approach involves first writing down the conjecture based on brane counting for general  $k$ , since the methods of [5] turn out to generalise from the  $k = 1$  to the general  $k$  case. The conjecture for general  $k$ , based on the brane counting, follow immediately from (A.5)

$$\frac{1}{N(k, n)}STr(X_i X_i)^m \Big|_{m=-k+\frac{1}{2}} \simeq \prod_{i=1}^{2k-1} \frac{1}{(n+i)} . \tag{A.12}$$

Note that the right hand side of this equation appears in the factor outside the sum in (A.3) and (A.4). Notice also that the above expression concerns the large  $n$  expansion of the symmetrised trace considered at  $m = -k + \frac{1}{2}$ .

One can repeat the  $k = 1$  calculation of [5] for general  $k$ , to check the first three terms of this conjecture. A sketch of this calculation follows before displaying the full results. First we calculate  $\frac{1}{N(k, n)}STr(X_i X_i)^m$  for  $m = 2, 3, 4$ . Then we find the first three terms in the symmetrised trace, large  $n$  expansion using these results. Finally we can check that the conjecture (A.12) is true for the first three terms in the symmetrised trace large  $n$  expansion, for general  $k$  as well as for  $k = 1$ . We then proceed to calculate the fourth term in the expansion, for general  $k$ . To do this we need to calculate  $\frac{1}{N(k, n)}STr(X_i X_i)^m$  for  $m = 5, 6$ . We then show that the fourth term in the large  $n$  expansion of the symmetrised trace agrees precisely with (A.12).

In the following, we use the notation of [5], with each trace of a string of  $2m$   $X_i$  matrices arising here being represented by a chord diagram with  $m$  chords. This provides a convenient way to represent equivalent strings of matrices.

For the calculation of  $STr(X_i X_i)^2$  there are three different strings of the four  $X_i$  matrices and two different chord diagrams. Two of the three strings correspond to the same chord diagram. In the following, the first column contains a fraction which is the multiplicity of the chord diagram in the list of strings divided by the total number of strings. The second column contains a picture of the chord diagram preceded by an example of a string in the equivalence class defined by this chord diagram. The evaluation of the chord diagram is the final entry.

$$\begin{aligned} \frac{2}{3} \ 1122 &= \text{Diagram 1} = c^2, \\ \frac{1}{3} \ 1212 &= \text{Diagram 2} = (c - 4k) \text{Diagram 3} = c(c - 4k). \end{aligned}$$

Using this, one finds immediately that

$$\frac{1}{N(k, n)} STr(X_i X_i)^2 = c^2 - \frac{4}{3}kc. \tag{A.13}$$

For  $m = 3$  there are 15 different strings of  $X_i$  matrices and five different chord diagrams, which evaluate as follows

$$\begin{aligned} \frac{2}{15} \ 112233 &= \text{Diagram 1} = c^3, \\ \frac{6}{15} \ 112323 &= \text{Diagram 2} = c \text{Diagram 3} = c^2(c - 4k), \\ \frac{3}{15} \ 112332 &= \text{Diagram 4} = c^3, \\ \frac{3}{15} \ 121323 &= \text{Diagram 5} = (c - 4k) \text{Diagram 6} = c(c - 4k)^2, \\ \frac{1}{15} \ 123123 &= \text{Diagram 7} = c^3 - 12kc^2 + 16k(k + 1)c. \end{aligned}$$

Thus we find that

$$\frac{1}{N(k, n)} STr(X_i X_i)^3 = c^3 - 4kc^2 + \frac{16}{15}k(4k + 1)c. \tag{A.14}$$

For  $m = 4$  there are 105 different strings and 18 different chord diagrams. We omit the details for simplicity. The final result is that

$$\frac{1}{N(k, n)} STr(X_i X_i)^4 = c^4 - 8kc^3 + \frac{16}{5}k(7k + 2)c^2 - \frac{64}{105}k(34k^2 + 24k + 5)c. \tag{A.15}$$

For  $m=5$  there are 945 different strings of  $X_i$  matrices, and 105 different chord diagrams.<sup>2</sup> The result is

$$\begin{aligned} \frac{1}{N(k, n)} STr(X_i X_i)^5 &= c^5 - \frac{40}{3} k c^4 + \frac{16}{3} (13k + 4) k c^3 - \frac{64}{63} (158k^2 + 126k + 31) k c^2 \\ &\quad + \frac{256}{945} (496k^3 + 672k^2 + 344k + 63) k c. \end{aligned} \tag{A.16}$$

For  $m=6$  there are 10395 different strings of  $X_i$  matrices, and 902 different chord diagrams, and we find that

$$\begin{aligned} \frac{1}{N(k, n)} STr(X_i X_i)^6 &= c^6 - 20k c^5 + \frac{16}{3} (31k + 10) k c^4 - \frac{64}{63} (677k^2 + 582k + 157) k c^3 \\ &\quad + \frac{256}{315} (1726k^3 + 2616k^2 + 1541k + 336) k c^2 \\ &\quad - \frac{1024}{10395} (11056k^4 + 24256k^3 + 22046k^2 + 9476k + 1575) k c. \end{aligned} \tag{A.17}$$

Now we calculate the first four terms in the large  $n$  expansion of  $STr(X_i X_i)^m$ . Suppose that the coefficient of  $c^{m-l}$  term in  $STr(X_i X_i)^m$  is a polynomial in  $m$  of order  $2l$ . Then we have the following ansatz: The known factors of these polynomials come from the fact that the series has to terminate so that there are never negative powers of  $c$  for  $m = 1, 2, 3, \dots$ . Then

$$\begin{aligned} \frac{1}{N(k, n)} STr(X_i X_i)^m &= c^m + y_1(k) m(m-1) c^{m-1} \\ &\quad + \left( y_2(k) m + y_3(k) \right) m(m-1)(m-2) c^{m-2} \\ &\quad + \left( y_4(k) m^2 + y_5(k) m + y_6(k) \right) m(m-1)(m-2)(m-3) c^{m-3} \\ &\quad + O(c^{m-4}). \end{aligned}$$

We now find the unknown functions  $y_1(k), y_2(k) \dots y_6(k)$  using the results of  $STr(X_i X_i)^m$  for  $m = 2, 3, 4, 5, 6$  calculated above. We find that

$$\begin{aligned} y_1(k) &= -\frac{2}{3} k, & y_2(k) &= \frac{2}{45} (5k + 2) k, \\ y_3(k) &= \frac{2}{45} (k - 2) k, & y_4(k) &= \frac{1}{2835} (-140k^2 - 168k - 64) k, \\ y_5(k) &= \frac{1}{2835} (-84k^2 + 216k + 192) k, & y_6(k) &= \frac{1}{2835} (128k^2 + 96k - 104) k. \end{aligned} \tag{A.18}$$

Now we are able to provide a check of the conjecture (A.12). First we express the right-hand side of (A.12) as a function of  $c$  as

$$\prod_{l=1}^{2k-1} \frac{1}{n+l} = \frac{1}{\sqrt{(k^2+c)}} \prod_{l=1}^{k-1} \frac{1}{c+2kl-l^2} = c^{-k+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{b_j}{c^j}, \tag{A.19}$$

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<sup>2</sup>We acknowledge the assistance of Simon Nickerson, for writing a computer programme used here. Maple files for these calculations are available from the authors.

where

$$\begin{aligned}
 b_1 &= -\frac{2}{3}k\left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right), \\
 b_2 &= \frac{1}{45}(10k^2 - 3k + 2)k\left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\left(k + \frac{3}{2}\right), \\
 b_3 &= \frac{1}{2835}(-24 + 34k - 61k^2 + 56k^3 - 140k^4) \times \\
 &\quad k\left(k - \frac{1}{2}\right)\left(k + \frac{1}{2}\right)\left(k + \frac{3}{2}\right)\left(k + \frac{5}{2}\right). \tag{A.20}
 \end{aligned}$$

Now consider the left-hand side of (A.12) involving the large  $n$  expansion of  $STr(X_i X_i)^m$ , which we calculated above, but now we set  $m = -k + \frac{1}{2}$ . Expanding in inverse powers of  $c$ , we find that this becomes

$$\frac{1}{N(k, n)} STr(X_i X_i)^m \Big|_{m=-k+1/2} = c^{-k+\frac{1}{2}} \sum_{j=0}^{\infty} \frac{b_j}{c^j},$$

with precisely the coefficients  $b_i$  given in (A.20). Given the extensive and non-trivial calculations required to obtain these results, we believe that there is strong evidence for the truth of (A.12).

For  $k = 1$  the guess of the exact answer for  $n$  even is

$$\frac{1}{N(1, n)} STr(X_i X_i)^m = \frac{2(2m+1)}{n+1} \sum_{i=1}^{\frac{n}{2}} (2i)^{2m}. \tag{A.21}$$

It is easy to show that (A.21) agrees with the first four orders in the large  $n$  expansion (A.18) for  $k = 1$ . If we set  $m = -\frac{1}{2}$  in this we get zero, because of the  $(2m+1)$  factor. This might appear to contradict (A.11), but it is easy to show, using a large  $n$  expansion, that if (A.21) is true then (A.11) holds to all orders. To calculate the large  $n$  expansion of this sum we can use the Euler-Maclaurin formula. This approximates the sum by an integral, plus an infinite series of corrections involving the Bernoulli numbers  $B_{2p}$

$$\begin{aligned}
 \sum_{i=1}^n f(i) &\simeq \int_0^{n+1} f(x) dx + \frac{1}{2}[f(n+1) - f(0)] \\
 &\quad + \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!} [f^{(2p-1)}(n+1) - f^{(2p-1)}(0)]. \tag{A.22}
 \end{aligned}$$

We see from this calculation that for  $k = 1$  the value  $m = -\frac{1}{2}$  is very special. It is the only value of  $m$  for which the higher order terms in the Euler-MacLaurin large  $n$  approximation of the sum in (A.21) are zero.

## B. Finite $n$ results on symmetrised traces from the highest weight method

Results on finite  $n$  symmetrised traces can be obtained by generalising the highest weight method of [5]. For the  $SO(3)$  representations used in fuzzy 2-spheres we have

$$\frac{1}{2} STr_{J=1/2}(\alpha_i \alpha_i)^m = (2m+1), \tag{B.1}$$

where the  $1/2$  comes from dividing with the dimension of the spin- $1/2$  representation. A similar factor will appear in all of the results below. The above result was derived in [5]. For the spin one case, we will obtain

$$\frac{1}{3} ST r_{J=1}(\alpha_i \alpha_i)^m = \frac{2^{2m+1}(2m+1)}{3}. \quad (\text{B.2})$$

These results can be generalised to representations of  $SO(2l+1)$  relevant for higher fuzzy spheres. The construction of higher dimensional fuzzy spheres uses irreducible representations of highest weight  $(\frac{n}{2}, \dots, \frac{n}{2})$ , as we have noted. For the minimal representation with  $n=1$  we have

$$\frac{1}{D_{n=1}} ST r_{n=1}(X_i X_i) = \frac{(2l+2m-1)!!}{(2m-1)!!(2l-1)!!}. \quad (\text{B.3})$$

Notice the interesting symmetry under the exchange of  $l$  and  $m$ . For the next-to-minimal irreducible representation with  $n=2$  we obtain:

$$\frac{1}{D_{n=2}} ST r_{n=2}(X_i X_i) = 2^{2m}(l+1) \frac{(2l+2m-1)!!}{(2m-1)!!(2l+1)!!}. \quad (\text{B.4})$$

This is a generalisation of the spin 1 case to higher orthogonal groups. It agrees with the formulae in section A of the appendix, with  $l \rightarrow k$ .

### B.1 Review of spin half for $SO(3)$

We will begin by recalling some facts about the derivation of the  $n=1$  case in [5]. The commutation relations can be expressed in terms of  $\alpha_3, \alpha_{\pm}$

$$\begin{aligned} \alpha_{\pm} &= \frac{1}{\sqrt{2}}(\alpha_1 \pm i\alpha_2), \\ [\alpha_3, \alpha_{\pm}] &= 2\alpha_{\pm}, \\ [\alpha_+, \alpha_-] &= 2\alpha_3, \\ c &= \alpha_+ \alpha_- + \alpha_- \alpha_+ + \alpha_3^2. \end{aligned} \quad (\text{B.5})$$

With these normalisations, the eigenvalues of  $\alpha_3$  in the spin half representation are  $\pm 1$  and  $\alpha_+ \alpha_-$  is 1 on the highest weight state.

It is useful to define a quantity  $\tilde{C}(p, q)$  which depends on two natural numbers  $p, q$  and counts the number of ways of separating  $p$  identical objects into  $q$  parts

$$\tilde{C}(p, q) = \frac{(p+q-1)!}{p!(q-1)!}. \quad (\text{B.6})$$

We begin by a review of the spin half case, establishing a counting which will be used again in more complicated cases below. This relies on a sum

$$2^k \sum_{\hat{i}_{2k}}^{2n-2k} \dots \sum_{\hat{i}_2=0}^{\hat{i}_3} \sum_{\hat{i}_1=0}^{\hat{i}_2} (-1)^{\hat{i}_1+\hat{i}_2+\dots+\hat{i}_{2k}} = 2^k \frac{n!}{(n-k)!k!}.$$

Recall that this sum was obtained by evaluating a sequence of generators of  $SO(3)$  consisting of  $k$  pairs  $\alpha_- \alpha_+$  and with powers of  $\alpha_3$  between these pairs -

$$\alpha_3^{2J_{2k+1}} \alpha_+ \alpha_3^{2J_k} \alpha_- \cdots \alpha_- \alpha_3^{J_3} \alpha_+ \alpha_3^{J_2} \alpha_- \alpha_3^{J_1} . \tag{B.7}$$

We can move the powers of  $\alpha_3$  to the left to get factors  $(\alpha_3 - 2)^{J_2+J_4+\cdots+J_{2k}}$ . Moving the  $\alpha_3$  with powers  $J_1, J_3 \dots$  gives  $\alpha_3^{J_1+J_3+\cdots}$ . The  $k$  powers of  $\alpha_- \alpha_+$  gives  $2^k$ . The above sum can be rewritten

$$2^k \sum_{J_{2k+1}=0}^{2m-2k} \cdots \sum_{J_2=0}^{2m-2k-J_3+\cdots+J_{2k+1}} \sum_{J_1=0}^{2m-2k-J_2+\cdots+J_{2k+1}} (-1)^{J_2+J_4+\cdots+J_{2k}} = 2^k \frac{n!}{(n-k)!k!} . \tag{B.8}$$

This includes a sum over  $J_e = J_2 + J_4 + \cdots + J_{2k}$ . The summand does not depend on the individual  $J_2, J_4, \dots$  only on the sum  $J_e$  which ranges from 0 to  $2m - 2k$ . The sum over  $J_2, J_4, \dots$  is the combinatoric factor, introduced above, which is the number of ways of splitting  $J_e$  identical objects into  $k$  parts, i.e.  $\tilde{C}(J_e, k)$ . The remaining  $2m - 2k - J_e$  powers of  $\alpha_3$  are distributed in  $k + 1$  slots in  $\tilde{C}(2m - 2k - J_e, k + 1)$  ways. Hence the sum (B.8) can be written more simply as

$$2^k \sum_{J_e=0}^{2m-2k} (-1)^{J_e} \tilde{C}(J_e, k) \tilde{C}(2m - 2k - J_e, k + 1) = 2^k \frac{m!}{(m-k)!k!} . \tag{B.9}$$

Then there is a sum over  $k$  from 0 to  $m$ , with weight

$$C(k, m) = \frac{2^k k! (2m - 2k)! m!}{(m - k)! (2m)!} \tag{B.10}$$

which gives the final result  $2m + 1$  [5]. Similar sums arise in the proofs below. In some cases, closed formulae for the sums are obtained experimentally.

## B.2 Derivation of symmetrised trace for minimal $SO(2l + 1)$ representation

The Casimir of interest here is

$$X_\mu X_\mu = X_{2l+1}^2 + \sum_{i=1}^l \left( X_-^{(i)} X_+^{(i)} + X_+^{(i)} X_-^{(i)} \right) . \tag{B.11}$$

The patterns are similar to those above, with  $\alpha_3$  replaced by  $X_{2l+1}$ , and noting that here there are  $l$  ‘‘colours’’ of  $\alpha_\pm$  which are  $X_\pm^{(l)}$ . All the states in the fundamental spinor are obtained by acting on a vacuum which is annihilated by  $l$  species of fermions. Generally we might expect patterns

$$\cdots X_{2l+1}^{i_1} X_-^{(j_1)} X_{2l+1}^{i_2} X_+^{(j_2)} \cdots \tag{B.12}$$

In evaluating these, we can commute all the  $X_{2l+1}$  to the left. This results in shifts which do not depend on the value of  $j$ . It is easy to see that whenever  $X_+^{(1)}$  is followed by  $X_+^{(1)}$  we get zero because of the fermionic construction of the gamma matrices.  $X_+^{(1)}$  cannot also



be followed by  $X_+^{(2)}$  because  $X_+^{(1)}X_+^{(2)} + X_+^{(2)}X_+^{(1)} = 0$ . So the pairs have to take the form  $X_-^{(j)}X_+^{(j)}$  for fixed  $j$ . The sum we have to evaluate is

$$\begin{aligned} & \sum_{k=0}^m \sum_{J_e=0}^{2m-2k} (-1)^{J_e} \tilde{C}(J_e, k) \tilde{C}(2m-2k-J_e, k) \tilde{C}(k, l) 2^k C(k, m) \\ &= \sum_{k=0}^m \binom{m}{k} 2^k C(k, m) \tilde{C}(k, l) \\ &= \frac{(2l+2m-1)!!}{(2m-1)!!(2l-1)!!} \end{aligned} \tag{B.13}$$

The factors  $\tilde{C}(J_e, k)$  and  $\tilde{C}(2m-2k-J_e, k)$  have the same origin as in the spin half case. The factor (B.10) is now generalised to an  $l$ -colour version

$$C(k_1, k_2 \dots k_l; m) = 2^k \frac{(2m-2k)!}{(2m)!} \frac{m!}{(m-k)!} k_1! k_2! \dots k_l! \tag{B.14}$$

This has to be summed over  $k_1, \dots, k_l$ . For fixed  $k = k_1 + \dots + k_l$  we have

$$\sum_{k_1 \dots k_l} C(k_1 \dots k_l, m) \frac{k!}{k_1! \dots k_l!} = \sum_{k_1 \dots k_l} C(k, m) = C(k, m) \tilde{C}(k, l) \tag{B.15}$$

The combinatoric factor  $\frac{k!}{k_1! \dots k_l!}$  in the second line above comes from the different ways of distributing the  $k_1 \dots k_l$  pairs of  $(-+)$  operators in the  $k$  positions along the line of operators. The subsequent sum amounts to calculating the number of ways of separating  $k$  objects into  $l$  parts which is given by  $\tilde{C}(k, l)$ . The  $C(k, m)$  is familiar from (B.10). This sum can be done for various values of  $k, m$  and gives agreement with (B.3).

### B.3 Derivation of spin one symmetrised trace for SO(3)

For the spin one case more patterns will arise. After an  $\alpha_-$  acts on the highest weight, we get a state with  $\alpha_3 = 0$  so that we have, for any positive  $r$

$$\alpha_3^r \alpha_- |J = 1, \alpha_3 = 2\rangle = 0, \quad \forall r > 0 \tag{B.16}$$

Hence any  $\alpha_-$  can be followed immediately by  $\alpha_+$ . These neutral pairs of  $(\alpha_+ \alpha_-)$  can be separated by powers of  $\alpha_3$ . Alternatively an  $\alpha_-$  can be followed immediately by  $\alpha_-$ . The effect of  $\alpha_-^2$  is to change the highest weight state to a lowest weight state. In describing the patterns we have written the ‘‘vacuum changing operator’’ on the second line, with the first line containing only neutral pairs separated by  $\alpha_3$ ’s. Let there be  $J_1$  neutral pairs in this first line and  $L_1$  powers of  $\alpha_3$  distributed between them. After the change of vacuum, we can have a sequence of  $(\alpha_- \alpha_+)$  separated by powers of  $\alpha_3$ . Let there be a total of  $J_2$  neutral pairs and  $L_2$   $\alpha_3$ ’s in the second line. At the beginning of the third line we have another vacuum changing operator  $\alpha_+^2$  which takes us back to the highest weight state. In the third line, we have  $J_3$  neutral pairs and  $L_3$  powers of  $\alpha_3$ . The equation below describes a general pattern with  $p$  pairs of vacuum changing operators. The total number of neutral



$$\begin{aligned}
X_{2l+1}^r X_- X_+ |0\rangle &= X_- X_+ X_{2l+1}^r |0\rangle = (2)^r X_- X_+ |0\rangle, \\
X_- X_+ X_{2l+1}^r X_+^2 |0\rangle &= 0, \quad Y_+ X_+^2 + X_+^2 Y_+ |0\rangle = 0, \\
X_+ Y_+ X_+ |0\rangle &= 0, \quad X_- Y_+ X_+ |0\rangle = 0, \\
X_+ X_- X_+^2 |0\rangle &= Q(2, 1) X_+^2 |0\rangle, \quad X_+ X_- X_+ Y_+ |0\rangle = Q(2, 1) X_+ Y_+ |0\rangle, \\
X_-^2 X_+^2 |0\rangle &= Q(2, 2) |0\rangle, \quad Y_- X_- X_+ Y_+ |0\rangle = Q(2, 2) |0\rangle.
\end{aligned}$$

It is significant that the same  $Q(2, 1), Q(2, 2)$  factors appear in the different places in the above equation. In the above  $X_+$  stands for any of the  $l$   $X_+^{(i)}$ 's. Any equation containing  $X_{\pm}$  and  $Y_{\pm}$  stands for any pair  $X_{\pm}^{(i)}$  and  $X_{\pm}^{(j)}$  for  $i, j$  distinct integers from 1 to  $l$ .

The general pattern is similar to (B.17) with the only difference that the  $(\alpha_- \alpha_+)$  on the first line is replaced by any one  $(X_-^{(i)} X_+^{(i)})$  for  $i = 1, \dots, l$ . The positive vacuum changing operators can be  $(X_+^{(i)} X_+^{(j)})$ , where  $i, j$  can be identical or different. For every such choice the allowed neutral pairs following them are  $X_+^{(j)} X_-^{(i)}$  and the dual vacuum changing operator is  $(X_-^{(j)} X_-^{(i)})$ .

The summation we have to do is:

$$\begin{aligned}
\sum_{p=0}^{[m/2]} \sum_{J=0}^{m-2p} \sum_{L_e=0}^{2m-4p-2J} \sum_{J_e=0}^J & \left( C(2p+J, m) \tilde{C}(2p+J, l) \tilde{C}(J_e, p) \tilde{C}(J-J_e, p+1) \times \right. \\
& (-1)^{L_e} \tilde{C}(L_e, J_e+p) \tilde{C}(2m-2J-4p-L_e, J-J_e+p+1) \times \\
& \left. 2^{2m-2J-4p} Q(1, 1)^{J-J_e} Q(2, 1)^{J_e} Q(2, 2)^p \right).
\end{aligned}$$

The  $Q$ -factors can be easily evaluated on the highest weight and then inserted into the above

$$Q(1, 1) = 4, \quad Q(2, 1) = 4 \quad Q(2, 2) = 16. \tag{B.19}$$

By computing this for several values of  $m, l$ , we obtain (B.4). Note that both the  $l = 1$  and the general  $l$  case will yield the correct value for  $m = 0$ , which is 1.

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